Tunnelling with Bianchi IX Instantons.

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Within the context of finding the initial conditions of the universe we consider gravitational instantons falling into the Bianchi IX classification. That is, a Euclidean four-manifold with a metric that satisfies Einstein's equations with an induced metric on S^3 submanifolds that is homogeneous but anisotropic. As well as finding regular solutions to the field equations with a tunnelling scalar field, we also look at the case of singular instantons with a view to applying the results to generic potentials. The study is in agreement with the prejudice that instantons with higher symmetry have a lower Euclidean action, even when we consider the singular class of solutions. It is also found that the Euclidean action can diverge for simple potentials, showing that the Hawking Turok instanton had finite action owing to its symmetry.

I. INTRODUCTION

Questions of initial conditions in the universe have been around for many years, but only now with the promise of detailed data from the cosmic microwave background can we expect significant experimental input. The proposal which this work is concerned with is that of the universe 'tunnelling from nothing'. That such a situation could be realised was noticed by Vilenkin [1], where he pointed out that the compact nature of the de Sitter instanton could be interpreted as the universe coming from nothing into a de Sitter phase via the de Sitter instanton. In a similar vein Hartle and Hawking [2] attempted to describe the initial conditions of our universe by proposing that the Lorentzian metric of spacetime be rounded off with a Euclidean manifold. The two approaches are similar in philosophy but different in details, leading to different probability measures for the initial conditions of the universe [3].

Studying instantons is at some level independent of these considerations. What one is looking for is a Euclidean solution of the field equations which allows a Lorentzian solution to be glued smoothly onto it, the question of probability measures comes later. As such, we start by just trying to find solutions to general relativity on a manifold with Riemannian metric, a daunting task as there are an infinite number of degrees of freedom in the system. To aid us we impose some symmetry requirements that leave us with a tractable problem, this is not entirely ad hoc as the analogous system of a field tunnelling in flat space shows. Coleman *et al* [4] showed that imposing SO(4) symmetry on a bubble at zero temperature was sensible in that it had the lowest action and was therefore more likely to nucleate. In fact it was later shown that it was more than just sensible, because all other solutions to such a system are singular [5].

The idea of singular solutions has been brought to the fore by Hawking and Turok [6]; there it was pointed out that whereas singular solutions in flat space means infinite action, this is not the case in gravity. Including gravity means that when the solution starts to blow up the manifold at least has a chance to close up and make the volume of the instanton finite, allowing a non divergent action. Whether such solutions are to be allowed into the path integral has been hotly debated, here we take the opinion that we should really let the action decide what is allowed rather than introducing some notion of singularity by hand to disallow these instantons. We shall find in fact that this is more restricting than may be expected. It was asserted in [6] that generic potentials lead to finite action for solutions with SO(4) symmetry (although this was later shown to be true only for polynomial and certain exponential potentials [7]). We will find here that even with polynomial potentials the symmetry of Bianchi IX is not always enough to keep the action regular, unlike imposing SO(4) symmetry.

II. EINSTEIN EQUATIONS.

In this paper we consider the field theory of general relativity coupled to a scalar field according to the following action,

$$S_{\rm E} = \int \eta \left[-\frac{1}{2\kappa}R + \frac{1}{2}(\partial\phi)^2 + \mathcal{V}(\phi)\right] + \text{boundary term}, \quad (1)$$

where the boundary terms are designed to cancel the second derivatives of the metric, allowing a sensible meaning to the variational principle on a boundary [8], [9]. The gravitational coupling is $\kappa = 8\pi G$, which we rescale to unity, and η is the volume four-form of the manifold. The metric is taken to be positive definite and of the form,

$$\mathrm{ds}^2 = \mathrm{d}\tau^2 + \mathrm{a}(\tau)^2(\sigma^1)^2 + \mathrm{b}(\tau)^2(\sigma^2)^2 + \mathrm{c}(\tau)^2(\sigma^3)^2. \quad (2)$$

The σ^i are the left invariant one forms of SU(2) and so satisfy $\mathrm{d}\sigma^{\rm i} = -\frac{1}{2}\epsilon^{\rm ijk}\sigma^{\rm j}\wedge\sigma^{\rm k}$, making the solution fall into the Bianchi IX class. Using Cartan's structure equations for a torsion free connection it is a nice exercise to find the curvature two forms, leading to the following field equations,

$$
\begin{aligned}
2\frac{a''}{a}+\frac{a'b'}{ab}+\frac{a'c'}{ac}-\frac{b'c'}{bc} &= -\frac{5}{4}\frac{a}{bc}\left(-\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right) \\
&+\frac{3}{4}\frac{b}{ac}\left(\frac{a}{bc}-\frac{b}{ac}+\frac{c}{ab}\right) \\
&+\frac{3}{4}\frac{c}{ab}\left(\frac{a}{bc}+\frac{b}{ac}-\frac{c}{ab}\right) \\
&-\frac{1}{2}\phi'^2-\mathcal{V}
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c} &= \frac{1}{4}\frac{a}{bc}\left(-\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right) \\
&+\frac{1}{4}\frac{b}{ac}\left(\frac{a}{bc}-\frac{b}{ac}+\frac{c}{ab}\right) \\
&+\frac{1}{4}\frac{c}{ab}\left(\frac{a}{bc}+\frac{b}{ac}-\frac{c}{ab}\right) \\
&+\frac{1}{2}\phi'^2-\mathcal{V}
\end{aligned} \tag{4}
$$

$$
\phi''+\frac{\beta'}{\beta}\phi'=\frac{\partial\mathcal{V}}{\partial\phi}. \tag{5}
$$

Here $\beta(\tau)=a(\tau)b(\tau)c(\tau)$ is the volume of the surface at constant τ and $'$ denotes differentiation with respect to τ. To get the equations of motion for $b(\tau)$ and $c(\tau)$ we use the $\mathbf{Z_3}$ symmetry of the equations and just cyclically permute a, b and c in (3).

III. THE REGULAR SOLUTIONS.

We know that for a potential with a false vacuum there is the Coleman-De Luccia solution [10] where we take the functions $a(\tau)$, $b(\tau)$ and $c(\tau)$ to be equal, the isotropic limit of Bianchi IX. When we want a regular solution we require it be regular everywhere, in particular at the origin. In the coordinates we are using one finds that the Coleman-De Luccia solution has the scale factor behaving as $a(\tau\to 0)\to\tau/2$ giving the instanton a metric which locally is like an isotropic metric on $\mathbf{R}^4$. This however is not the only way to smoothly end an instanton. The Coleman-De Luccia solution ends on what Gibbons and Hawking [11] called a nut, where all the scale factors vanish linearly as $\frac{1}{2}\tau$. There is also the case of closing the instanton off with a bolt, in this case where one of the scale factors vanishes linearly as $\frac{1}{2}\tau$ the other two become equal and constant. One could also consider the situation where the vanishing scale factor does so with unit proportionality to τ, this changes the group orbits of the homogeneous submanifolds to SO(3) rather than SU(2). Now we consider the regular bolt solution with SU(2) orbits on the homogeneous submanifolds, calling the point at $\tau=0$ the south pole. At the bolt then we impose that c varies as $\frac{1}{2}\tau$ and a, b are equal and constant, with the equations of motion showing that they remain equal.

A nice geometrical picture exists for these instantons in terms of expanding and contracting three spheres. The constant τ slices are seen to be homogeneous three spheres, whose anisotropy is dictated by the scale factor at that time. The nut-nut solution is viewed as an isotropic three sphere expanding from zero size at the south pole and contracting back to zero at the north pole. Looking at the bolt-bolt case we see that the ratio of the unequal scale factors represents the anisotropy of the S^3 slices, which diverges at the poles giving the two sphere limit of a squashed three sphere. The picture we have then is that at the south pole we have a maximally squashed S^3, becoming less squashed as we move off the pole, finally becoming the fully squashed S^2 limit at the north pole.

With the three volume β increasing linearly from zero at the south pole bolt we may infer from the Euclidean Raychaudhuri equation [12] that for some finite $\tau>0$ β will again go to zero, assuming that the scalar potential is positive semi-definite. The point at which the three volume goes back to zero we shall call the north pole. The manner in which it closes, i.e. singular or not, depends on the value of the constant that $a(0)\,(=b(0))$ is chosen to take, as well as on the value $\phi(0)$. We can see that $\phi(0)$ is not fixed by the constraint equation (4) by noting that near the bolt $a\to\chi+\xi\tau^2$, with χ and ξ arbitrary. The constraint equation then relates $\phi(0)$ to these arbitrary constants. Given that the 3 volume, β, vanishes and we are looking for regular $\phi(\tau)$ then (5) shows us that $\phi'=0$ at the north pole, as β'/β diverges there.

A similar argument to the undershot-overshoot argument of [13] suggests that values of $\phi(0)$ and $a(0)$ exist such that the instanton ends on a bolt at the north pole, if we have a potential with a false vacuum at $\phi=0$ and true vacuum for some positive ϕ. For a given $a(0)$ we may take $\phi(0)$ to be close to the vacuum value, in which case ϕ will diverge to negative values as we approach the north pole. If we start further from the vacuum value then ϕ will have turned around by the time the damping force β'/β of (5) becomes anti damping, driving ϕ to positive infinity. There is thus a value between these which make the scalar field constant at the north pole. We then find however that $a(0)$ has not been fixed yet. At the north pole the functions are now regular, however it will not in general be a bolt, with $c(\tau\to\tau_{\rm N})$ going like $\gamma(\tau-\tau_{\rm N})$ and γ being different from a half. We now use the freedom in the value of the constant that $a(0)$ takes to find the solution with $\gamma=\frac{1}{2}$.

To make these arguments concrete we took the potential

$$
\mathcal{V}(\phi)=(\phi^2-1)^2(\phi^2+\alpha), \tag{6}
$$

which exhibits a false vacuum at the origin for $0<\alpha<\frac{1}{2}$, we took $\alpha=0.1$. A regular solution using this potential has been found numerically and is displayed below in Fig. 1. We see from the profile of $\phi(\tau)$ that the scalar field interpolates between the false and true vacua, as in the Coleman-De Luccia case. However in that case one is able to find a section of the instanton which has zero extrinsic curvature, allowing one to use this surface to attach a Lorentzian metric to it corresponding to an open universe. Here however we do not expect to find such a three-surface owing to the anisotropy of the homogeneous submanifolds. We can however find a solution which has an inversion isometry, implying that the stable surface of this symmetry has zero extrinsic curvature [14]. Such a solution is shown in Fig. 2 where we see that the profiles are symmetric about the mid point between the north and south poles. Such an instanton nucleates a closed universe with anisotropic spatial sections.

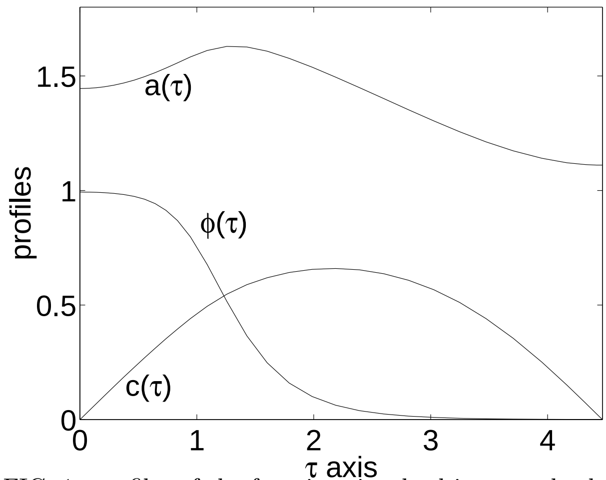

FIG. 1. profiles of the functions involved in a regular bolt instanton.

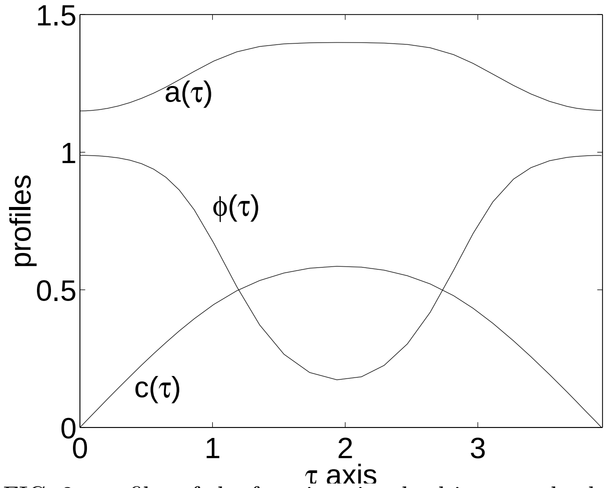

FIG. 2. profiles of the functions involved in a regular bolt instanton with a parity isometry.

IV. THE ‘CONICAL’ SOLUTIONS

The suggestion that singular solutions are valid allows us to consider the case where the metric functions are regular but the curvature diverges, reminiscent of the curvature blowing up at the end of a cone. This option was not available in the isotropic case, as the constraint (4) fixed the gradient of the metric functions. The freedom of the bolt type boundary conditions allows us to satisfy the constraint with $c(\tau)$ varying linearly with any gradient and $a(\tau)\,(= b(\tau))$ being an arbitrary constant. Here we consider such a case, imposing the condition that the instanton have a parity isometry such that we may attach a real Lorentzian metric to the stationary surface. On this surface the scale factors a and c take some constant value and their ratio, $\zeta = a(\tau_{\rm sym})/c(\tau_{\rm sym})$, gives a measure of the anisotropy, such that the isotropic case has ζ equal to unity. We then expect that varying ζ will allow us to move continuously from the bolt-bolt instanton to the Coleman-De Luccia solution with parity isometry. This is indeed the case and is illustrated below in Figs. 3, 4. The upper set of curves in Fig. 3 are the $a(\tau)$ scale factors as we vary the anisotropy measure, ζ, whilst the lower set are the $c(\tau)$. We see that one is able to interpolate between the two regular solutions (i.e. the Coleman-De Luccia case with $a(\tau) = c(\tau)$ and the bolt-bolt of Fig. 2) by using the solutions which have a conical singularity. To make the comparison clearer in this figure we have scaled all the solutions to have the same size, whereas in actual fact the instantons become larger as we approach the isotropic limit. This information is given in Fig. 4, where we can see how the size, τ_N, and the Euclidean action (including boundary terms) varies with anisotropy. One can see that the action is lowest for the isotropic case, confirming naive expectations. We also see that the bolt-bolt instanton has no privileged position, the anisotropy of the bolt-bolt case is around $\zeta = 2.4$ at which point the action shows no special behaviour. We also note here that the dotted line corresponds to the, separately calculated, nun-nut (Coleman-De Luccia) case.

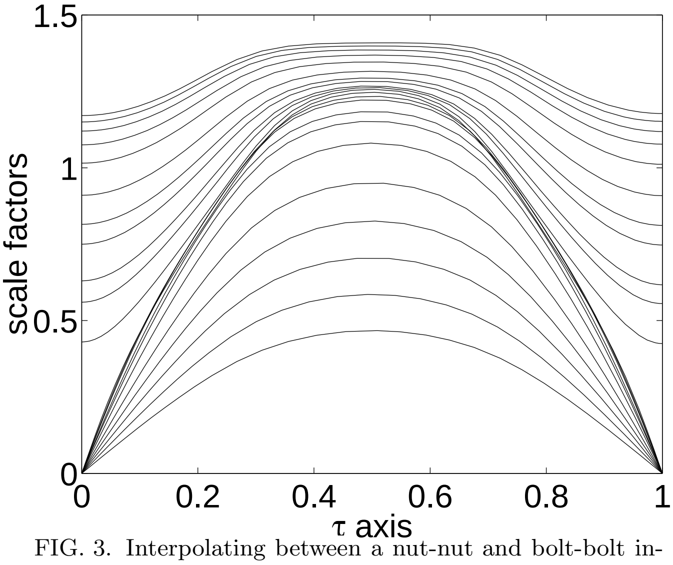

FIG. 3. Interpolating between a nut-nut and bolt-bolt instanton. The upper set being the a scale factor and the lower being the c.

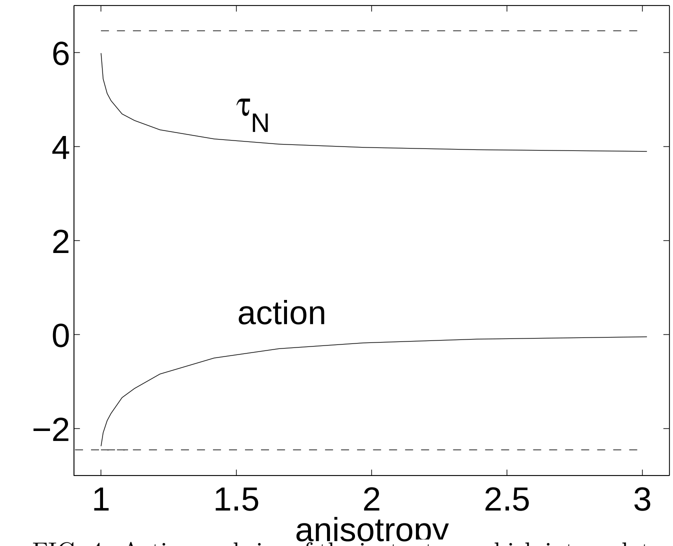

FIG. 4. Action and size of the instantons which interpolate between the nut-nut and bolt-bolt solution.

V. THE SINGULAR SOLUTIONS.

We now move on to consider the solutions which possess a singularity on the manifold different from the conical type discussed above. The type of singularity we are considering here is the analogue of the singularity studied in the Hawking Turok paper [6]. This singularity is different in that the curvature here diverges as $1/(\tau - \tau_{\rm N})$ whilst the conical singularity is more like a delta function at the poles.

As we are not looking for regular solutions we drop the tunnelling form of the potential used before and work with a simple quadratic potential. First we look for solutions which contain a reflection isometry. This is achieved by setting initial conditions at some point such that the fields a, b, c and ϕ all have vanishing gradient at that point, this allows a closed anisotropic universe to be grafted on to this surface.

We will be interested in finding the Euclidean action for such instantons and as such we need to take care of the boundary terms mentioned in (1). On our solution the singularity must be excised from the manifold, this means taking a small region around the singularity and discarding it, naturally introducing a boundary. The boundary term is then the contribution this boundary makes as the region is made arbitrarily small. We find that the Riemann curvature for the metric (2) is,

$$\begin{aligned} R = &-2(a''/a + b''/b + c''/c) - 2(a'/a + b'/b + c'/c) \quad (7)\\ &+\frac{1}{4}\frac{a}{bc}\left(-\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)\\ &+\frac{1}{4}\frac{b}{ac}\left(\frac{a}{bc}-\frac{b}{ac}+\frac{c}{ab}\right)\\ &+\frac{1}{4}\frac{c}{ab}\left(\frac{a}{bc}+\frac{b}{ac}-\frac{c}{ab}\right), \end{aligned}$$

giving the boundary term of (1) as,

$$\text{boundary term} = \frac{1}{\kappa}\left(\beta'(\tau \to \tau_S) - \beta'(\tau \to \tau_N)\right) V_3 \quad (8)$$

$$V_3 = \int \sigma^1 \wedge \sigma^2 \wedge \sigma^3. \quad (9)$$

In order to calculate with such ill behaved solutions one needs to be confident of the numerics. To be sure that the correct answers are at least well modelled we ran a number of simulations with different integrators. The results presented below are those calculated using a Bulirsch Stoer method [15], which was tested against a Runge Kutta routine also from [15] and the integrators found in mathematica and maple. All gave the same results to fractions of a percent. What one finds in doing this is that this simple potential does not always give finite action, as it did for the O(4) invariant solution. Following [6] we assume that near the singularity the form of the potential becomes irrelevant, a fact backed up by numerical solutions with polynomial potentials. The equation for ϕ then shows that near the north pole $\phi'(\tau \to \tau_N) \to 1/\beta$. Taking the constraint equation (4) assuming a form of $a \to (\tau_N - \tau)^A$, $b \to (\tau_N - \tau)^B$, $c \to (\tau_N - \tau)^C$, where A, B and C are positive, we then drop the terms which are products of scale factors. Using the fact that ϕ' is varying inversely as the three volume one finds $A + B + C = 1$, showing that dropping the products of scale factors was consistent. The three volume, β, therefore goes to zero linearly, implying that the scalar field diverges logarithmically, which is slow enough that the action can converge.

When looking at the action as a function of initial values for a, b and c we found the very striking result that unless two of the scale factors were equal the boundary term would diverge. Inspection of the numerical solution revealed that for such solutions not all of the scale factors went to zero at the singularity, this effect is coming about because it is the scale factor products, rather than ϕ' which dominates in the singularity. In fact such behaviour can also exist when two of the scale factors are equal, say a and b. It is found that the boundary term diverges unless $c(\tau = 0) < a(\tau = 0)$. The picture we then arrive at is given below in Fig. 5, which is a plot of the total action, when it is finite, for a set of singular instantons.

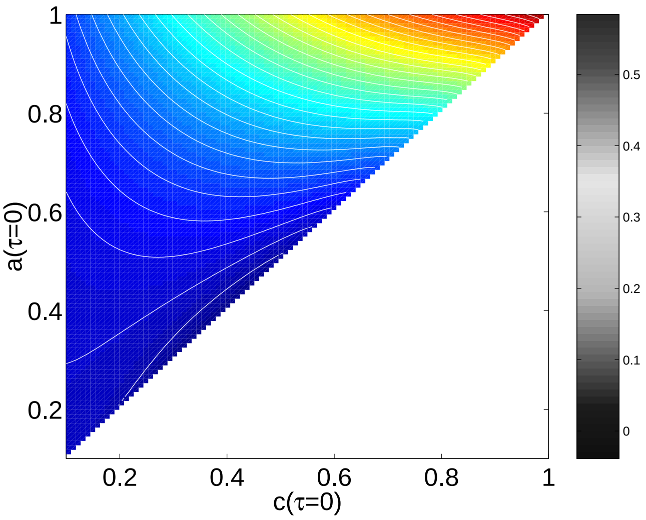

FIG. 5. Total action for the singular instanton with reflection isometry and $b(\tau) = a(\tau)$. Only the finite action solutions are shown.

Although this figure only covers the $b(\tau) = a(\tau)$ plane, it appears from the numerical results that it contains all the relevant information. The $\mathbf{Z_3}$ permutation symmetry of the scale factors mean that this plane is just the same as the $b(\tau) = c(\tau)$ and $a(\tau) = c(\tau)$ planes, with the action diverging off these planes. We then note that the minima of the action is to be found along the isotropy axis, that is to say the metric which has the O(4) isometry and is the gondola instanton studied in [16].

VI. CONCLUSIONS

We have discussed a generalisation of the SO(4) symmetric instantons, keeping the homogeneity assumption but relinquishing isotropy. In particular, our solutions were of the Bianchi IX type. It was found that the singular instantons in this class which allowed a reflection isometry, leading to the initial conditions of a closed universe, had a minima of the action when the instanton was isotropic. Moreover, the probability distribution had a $\mathbf{Z_3}$ symmetry around the axis of isotropy so the average instanton would lie on this axis (both mean and mode).

An in depth search of types of potential has not been carried out here, concentrating only on the simplest of potentials. Indeed, the results of this study did not reveal any reason to expect other potentials to deviate from the behaviour found here. One of the main results is that even for the singular instantons isotropy would seem to be preferred over anisotropy. The study also sheds light on the issue of whether or not to allow singular instantons, in that the action can in fact be a good discriminator of solutions, which may mean that removing solutions by hand is not necessary. Namely, we saw that it was the high degree of symmetry in the Hawking Turok solution that meant the criterion of finite action would keep all solutions with polynomial potentials.

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